

Comparing various commutation schemes for a sinusoidal motor

For all these cases, assume the individual winding back-EMF's to be of the form:

$$\varepsilon_A = k_v \dot{\theta} \sin(\theta) \quad (1.1)$$

$$\varepsilon_B = k_v \dot{\theta} \sin(\theta + 120^\circ) \quad (1.2)$$

$$\varepsilon_C = k_v \dot{\theta} \sin(\theta + 240^\circ) \quad (1.3)$$

And the individual winding torques are given by:

$$T_A = k_t i_A \sin(\theta) \quad (1.4)$$

$$T_B = k_t i_B \sin(\theta + 120^\circ) \quad (1.5)$$

$$T_C = k_t i_C \sin(\theta + 240^\circ) \quad (1.6)$$

Sinusoidal Commutation with a sinusoidal motor

Assume the three winding currents are given by:

$$i_A = I \sin(\theta) \quad (1.7)$$

$$i_B = I \sin(\theta + 120^\circ) \quad (1.8)$$

$$i_C = I \sin(\theta + 240^\circ) \quad (1.9)$$

The individual winding torques are then given by:

$$T_A = k_t i_A \sin^2(\theta) \quad (1.10)$$

$$T_B = k_t i_B \sin^2(\theta + 120^\circ) \quad (1.11)$$

$$T_C = k_t i_C \sin^2(\theta + 240^\circ) \quad (1.12)$$

The total torque produced is the sum of these three:

$$\begin{aligned} T &= T_A + T_B + T_C \\ &= k_t I \left[\sin^2(\theta) + \sin^2(\theta + 120^\circ) + \sin^2(\theta + 240^\circ) \right] \\ &= \frac{3}{2} k_t I \end{aligned} \quad (1.13)$$

where we've used the identity:

$$\sin^2(\theta) + \sin^2(\theta + 120^\circ) + \sin^2(\theta + 240^\circ) = \frac{3}{2} \quad (1.14)$$

Rewriting the torque again:

$$\boxed{T = \frac{3}{2} k_t I} \quad (1.15)$$

Constant-Current Trapezoidal Commutation

With this scheme, the winding currents are of the form:

$$i_A = \begin{cases} +i_0 & ; & 30^\circ \leq \theta < 150^\circ \\ 0 & ; & 150^\circ \leq \theta < 210^\circ, 330^\circ \leq \theta < 390^\circ \\ -i_0 & ; & 210^\circ \leq \theta < 330^\circ \end{cases} \quad (1.16)$$

$$i_B = \begin{cases} +i_0 & ; & -90^\circ \leq \theta < 30^\circ \\ 0 & ; & 30^\circ \leq \theta < 90^\circ, 210^\circ \leq \theta < 270^\circ \\ -i_0 & ; & 90^\circ \leq \theta < 210^\circ \end{cases} \quad (1.17)$$

$$i_C = \begin{cases} +i_0 & ; & 150^\circ \leq \theta < 270^\circ \\ 0 & ; & 90^\circ \leq \theta < 150^\circ, 270^\circ \leq \theta < 330^\circ \\ -i_0 & ; & -30^\circ \leq \theta < 90^\circ \end{cases} \quad (1.18)$$

The average torque produced over one of the six commutation states is given by:

$$\begin{aligned} \hat{T} &= \frac{3}{\pi} \int_{30^\circ}^{90^\circ} T_A + T_B + T_C d\theta \\ &= \frac{3}{\pi} \int_{30^\circ}^{90^\circ} k_t i_A \sin(\theta) + k_t i_B \sin(\theta + 120^\circ) + k_t i_C \sin(\theta + 240^\circ) d\theta \\ &= \frac{3}{\pi} \int_{30^\circ}^{90^\circ} k_t i_0 \sin(\theta) - k_t i_0 \sin(\theta + 240^\circ) d\theta \\ &= \frac{3k_t i_0}{\pi} \int_{30^\circ}^{90^\circ} \sin(\theta) - \sin(\theta + 240^\circ) d\theta \\ &= \frac{3k_t i_0}{\pi} \left[-\cos(\theta) + \cos(\theta + 240^\circ) \right]_{30^\circ}^{90^\circ} \\ &= \frac{3k_t i_0}{\pi} \left[\cos(30^\circ) - \cos(90^\circ) + \cos(330^\circ) - \cos(270^\circ) \right] \\ &= \frac{3k_t i_0}{\pi} \left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \right) = \frac{3\sqrt{3}k_t i_0}{\pi} \end{aligned} \quad (1.19)$$

Note that brushless DC motors are normally marketed with the assumption of being driven using trapezoidal commutation. For this reason, motor datasheets specify an “average” torque constant which is related to k_t by:

$$\hat{k}_t = \frac{3\sqrt{3}}{\pi} k_t \quad (1.20)$$

Rewriting the average motor torque in terms of each of these torque constants:

$$\boxed{\hat{T} = \frac{3\sqrt{3}}{\pi} k_t i_0 = \hat{k}_t i_0} \quad (1.21)$$

Constant-Voltage Trapezoidal Commutation

With this scheme, the winding currents in one of the six commutation states are of the form:

$$V_0 \equiv V_A - V_B$$

$$i_A = -i_B = \frac{V_0 - (\varepsilon_A - \varepsilon_B)}{R_{p-p}}$$

$$\begin{aligned} \varepsilon_A - \varepsilon_B &= k_v \dot{\theta} \left[\sin(\theta) - \sin(\theta + 120^\circ) \right] \\ &= k_v \dot{\theta} \left[\sin(\theta) - \sin(120^\circ) \cos(\theta) - \cos(120^\circ) \sin(\theta) \right] \\ &= k_v \dot{\theta} \left[\sin(\theta) - \frac{\sqrt{3}}{2} \cos(\theta) + \frac{1}{2} \sin(\theta) \right] \\ &= k_v \dot{\theta} \left[\frac{3}{2} \sin(\theta) - \frac{\sqrt{3}}{2} \cos(\theta) \right] \\ &= k_v \dot{\theta} \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \sin \left[\theta + \tan^{-1} \left(\frac{-\frac{\sqrt{3}}{2}}{\frac{3}{2}} \right) \right] \\ &= k_v \dot{\theta} \sqrt{3} \sin \left[\theta + \tan^{-1} \left(\frac{-1}{\sqrt{3}} \right) \right] \\ &= k_v \dot{\theta} \sqrt{3} \sin(\theta - 30^\circ) \end{aligned}$$

Plugging this into ...

$$i_A = -i_B = \frac{V_0 - k_v \dot{\theta} \sqrt{3} \sin(\theta - 30^\circ)}{R_{p-p}}$$

From 90 degrees to 150 degrees:

$$\begin{aligned}
\hat{T} &= \frac{3}{\pi} \int_{90^0}^{150^0} T_A + T_B d\theta \\
&= \frac{3}{\pi} \int_{90^0}^{150^0} k_t i_A \sin(\theta) + k_t i_B \sin(\theta + 120^0) d\theta \\
&= \frac{3}{\pi} \int_{90^0}^{150^0} k_t i_A [\sin(\theta) - \sin(\theta + 120^0)] d\theta \\
&= \frac{3}{\pi} \int_{90^0}^{150^0} k_t i_A \sqrt{3} \sin(\theta - 30^0) d\theta
\end{aligned}$$

Plugging in...

$$\begin{aligned}
\hat{T} &= \frac{3}{\pi} \int_{90^0}^{150^0} k_t \left[\frac{V_0 - k_v \dot{\theta} \sqrt{3} \sin(\theta - 30^0)}{R_{p-p}} \right] \sqrt{3} \sin(\theta - 30^0) d\theta \\
&= \frac{3k_t}{\pi R_{p-p}} \int_{90^0}^{150^0} [V_0 - k_v \dot{\theta} \sqrt{3} \sin(\theta - 30^0)] \sqrt{3} \sin(\theta - 30^0) d\theta \\
&= \frac{3k_t}{\pi R_{p-p}} \int_{90^0}^{150^0} V_0 \sqrt{3} \sin(\theta - 30^0) - 3k_v \dot{\theta} \sin^2(\theta - 30^0) d\theta \\
&= \frac{3k_t}{\pi R_{p-p}} \left\{ V_0 \sqrt{3} \int_{90^0}^{150^0} \sin(\theta - 30^0) d\theta - 3k_v \dot{\theta} \int_{90^0}^{150^0} \sin^2(\theta - 30^0) d\theta \right\} \\
&= \frac{3k_t}{\pi R_{p-p}} \left\{ V_0 \sqrt{3} \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \sin\left(\theta - \frac{\pi}{6}\right) d\theta - 3k_v \dot{\theta} \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \sin^2\left(\theta - \frac{\pi}{6}\right) d\theta \right\}
\end{aligned}$$

$$\begin{aligned}
\hat{T} &= \frac{3k_t}{\pi R_{p-p}} \left\{ V_0 \sqrt{3} \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \sin\left(\theta - \frac{\pi}{6}\right) d\theta - 3k_v \dot{\theta} \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \sin^2\left(\theta - \frac{\pi}{6}\right) d\theta \right\} \\
&= \frac{3k_t}{\pi R_{p-p}} \left\{ V_0 \sqrt{3} \left[-\cos\left(\theta - \frac{\pi}{6}\right) \right] \Big|_{\frac{\pi}{2}}^{\frac{5\pi}{6}} - 3k_v \dot{\theta} \frac{1}{2} \left[-\sin\left(\theta - \frac{\pi}{6}\right) \cos\left(\theta - \frac{\pi}{6}\right) + \theta \right] \Big|_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \right\} \\
&= \frac{3k_t}{\pi R_{p-p}} \left\{ V_0 \sqrt{3} \left[\cos\left(\frac{3\pi}{6} - \frac{\pi}{6}\right) - \cos\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) \right] - \frac{3k_v \dot{\theta}}{2} \left[\sin\left(\frac{\pi}{3}\right) \cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{2\pi}{3}\right) \cos\left(\frac{2\pi}{3}\right) + \frac{\pi}{3} \right] \right\} \\
&= \frac{3k_t}{\pi R_{p-p}} \left\{ V_0 \sqrt{3} \left[\cos\left(\frac{\pi}{3}\right) - \cos\left(\frac{2\pi}{3}\right) \right] - \frac{3k_v \dot{\theta}}{2} \left[\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} + \frac{\pi}{3} \right] \right\} \\
&= \frac{3k_t}{\pi R_{p-p}} \left\{ V_0 \sqrt{3} - \frac{3k_v \dot{\theta}}{2} \left[\frac{\sqrt{3}}{2} + \frac{\pi}{3} \right] \right\} \\
&= \frac{3\sqrt{3}k_t}{\pi} \frac{V_0}{R_{p-p}} - \frac{9k_t k_v \dot{\theta}}{2\pi R_{p-p}} \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3} \right) \\
\hat{T} &= \frac{3\sqrt{3}k_t}{\pi} \frac{V_0}{R_{p-p}} - \frac{9k_t k_v \dot{\theta}}{2\pi R_{p-p}} \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3} \right) \\
&= \frac{3\sqrt{3}k_t}{\pi} \left[\frac{V_0}{R_{p-p}} - \frac{3k_v \dot{\theta}}{2R_{p-p}} \left(\frac{1}{2} + \frac{\pi}{3\sqrt{3}} \right) \right]
\end{aligned}$$

Relating the Different Commutation Methods

We can use (average) total output torque to compare these three schemes. Equating torque for the first two schemes, we determine the equivalent constant-current trapezoidal current, i_0 , in terms of the sinusoidal commutation current, I :

$$\frac{3}{2}k_t I = \frac{3\sqrt{3}k_t i_0}{\pi} \Rightarrow i_0 = \frac{\pi}{2\sqrt{3}} I \quad (1.22)$$